Finite difference solution of a two-dimensional mathematical model of the cochlea

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A current, linear, two-dimensional mathematical model of the mechanics of the cochlea is solved numerically by using a finite difference approximation of the model equations. The finite-difference method is used to discretize Laplace's equation over a rectangular region with specified boundary conditions. The resulting matrix equation for fluid pressure is solved by using a Gaussian block-elimination technique. Numerical solutions are obtained for fluid pressure and basilar membrane displacement as a function of distance from the stapes. The finite difference method is a direct, versatile, and reasonably efficient means of solving the two-dimensional cochlear model.

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INTRODUCTION

The human cochlea is a spiraling tunnel through the temporal bone (with a radius of about 1 mm and a length of about 35 mm), where mechanical vibrations representing sound information are transduced into neural impulses. Experimental observation of the motion of the cochlear partition is difficult due to the small amplitude of vibration and the inaccessibility of the cochlea. Numerical solutions for a two-dimensional model of the cochlea have been reported by Lesser and Berkley (1972), Allen (1977), Allen and Sondhi (1979), Steele and Taber (1979), and Viergever (1980). This paper presents a straightforward and efficient numerical procedure that has not been previously applied to a cochlear model [Neely, (1978)].

The foundation of this paper is a simple two-dimensional mathematical model of the cochlea based on classical assumptions. A finite difference approximation of the model equations is used to obtain solutions for fluid pressure and displacement of the cochlear partition. Numerical solutions are obtained for displacement of the basilar membrane as a function of (1) distance from the stapes for selected frequencies and (2) frequency for selected distances from the stapes. The validity of finite difference solutions are supported by comparison with Allen's integral equation solution and by demonstrating the insensitivity of the finite difference solutions to the number of points used in the discretization. The objective of this paper is to demonstrate the practicality of solving the two-dimensional model equations directly by the finite difference method.

I. THE TWO-DIMENSIONAL MODEL EQUATIONS

The two-dimensional mathematical model of the cochlea presented in this section has evolved through the work of many investigators including: Ranke (1950), Peterson and Bogert (1950), Lesser and Berkley (1972), Lien (1973), Siebert (1974), Viergever and Kalker (1975), (1977), Geisler (1976), and Allen (1977). The model represents the cochlea as a rectangular region filled with an inviscid, incompressible fluid. This region is divided into two symmetric compartments by an elastic partition, which will be referred to as the basilar membrane (see Fig. 1). Stapes motion sets the cochlear fluid into motion and causes a vibratory deformation of the basilar membrane. All motion in this model is assumed to be linear to permit consideration of solutions in the frequency-domain. The properties of the basilar membrane are represented by an acoustic admittance function with no longitudinal coupling; thus the fluid provides the only means for longitudinal wave propagation. For futher discussion of these assumptions the reader is referred to the publications cited above.

For sinusoidal excitations to this linear, time-invariant model, the fluid pressure and motion can be expressed as complex functions of position representing the amplitude and phase of their sinusoidal time variation. Let $P_d(x,y)$ be a complex function which represents the pressure difference between the scalar tympani and the scala vestibuli:

$$P_d(x,y)=P_{sv}(x,y)-P_{st}(x,-y),$$

where $P_{sv}(x,y)$ is the total pressure in the scala vestibuli and $P_{st}(x,y)$ is the total pressure in the scala tympani. It is assumed that there is no variation in pressure in the z dimension and a time dependence of $\exp(i\omega t)$ is implied, where $\omega=2\pi f$ is the angular frequency in rad/s. (In Fig. 1 the z dimension would be

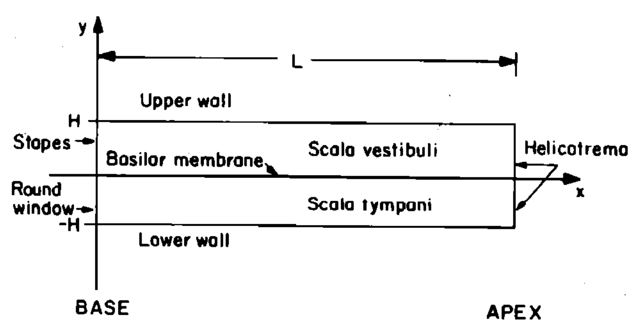

FIG. 1. Two-dimensional representation of the cochlea. Two fluid-filled canals, the scala tympani and the scala vestibuli, are bounded by an upper and lower rigid wall and separated by the basilar membrane. The height of each canal is H and its length is L. The stapes and round window at the base vibrate with equal, but opposite, amplitudes. The helicotrema at the apex provides a connecting passage between the two scalae.

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perpendicular to the page.) The pressure-difference function is required to satisfy Laplace's equation in the fluid:

$$\frac{\partial^2}{\partial x^2} P_d(x,y) + \frac{\partial^2}{\partial y^2} P_d(x,y) = 0, \quad 0<x<L, \quad 0<y<H. \tag{1}$$

The upper-wall boundary is assumed to be perfectly rigid, i.e., this boundary has no motion. The upper-wall boundary condition is

$$\frac{\partial}{\partial y} P_d(x,H) = 0, \quad 0<x<L. \tag{2}$$

The basilar membrane boundary is assumed to have (an unknown) motion in the y direction. The basilar membrane boundary condition is

$$\frac{\partial}{\partial y} P_d(x,0) = 2\rho a_b(x), \quad 0<x<L, \tag{3}$$

where $a_b(x)$ is the acceleration of the basilar membrane (defined to be positive for downward acceleration) and ρ is the volume density of the fluid. The basal-end boundary represents the stapes and the round window and is assumed to have (a known) motion in the x direction. The stapes and round window are assumed to move identically except in opposite directions. The basal boundary condition is

$$\frac{\partial}{\partial x} P_d(0,y) = -2\rho a_s, \quad 0<y<H, \tag{4}$$

where a_s is the acceleration of the stapes (constant in the y dimension and defined to be positive for inward acceleration). The apical-end boundary represents the helicotrema and has no pressure difference across it. The apical boundary condition is

$$P_d(L,y) = 0, \quad 0<y<H. \tag{5}$$

The apical boundary condition was chosen to be consistent with Allen (1977). It might be better to represent the helicotrema in line with the cochlear partition and to make the apical end rigid. However, the solutions to the model are not sensitive to the choice of apical boundary condition for frequencies above 400 Hz [see Viergever, (1980)].

In this paper, we will consider only sinusoidal excitation of the stapes at various frequencies. The stapes acceleration is chosen

$$a_s = -\omega^2 \tag{6}$$

in order to maintain constant stapes displacement (1 cm) for all frequencies. Since we have assumed there to be no longitudinal coupling in the basilar membrane, its properties can be represented by an acoustic admittance function $Y(x)$ which relates pressure difference to acceleration.

$$a_b(x) = i\omega Y(x) P_d(x,0). \tag{7}$$

We assume that $Y(x)$ has a simple second-order form

$$Y(x) = [K(x)/i\omega + R(x) + i\omega M(x)]^{-1}, \tag{8}$$

where $K(x)$, $R(x)$, and $M(x)$ are the stiffness, damping, and mass of the basilar membrane at position x, all defined per unit area (see Table I for physical units). The primary purpose of this model is to be able to solve for the displacement of the basilar membrane $D(x)$ for a given stapes motion. When Eqs. (1)–(5) have been solved, the basilar membrane displacement will be

$$D(x) = -a_b(x)/\omega^2. \tag{9}$$

II. DESCRIPTION OF THE NUMERICAL PROCEDURE

In order to find solutions to the model equations, a two-dimensional finite-difference approximation was used to discretize the x and y dimensions [Weinberger, 1965)]. Efficient solution of the resultant matrix equation required taking advantage of the special structure of a large matrix. In order to visualize this structure, it is helpful to consider the discretizing process in two steps.

First, we discretize the y dimension and set up a vector differential equation in x. The pressure-difference vector (with M components) will be defined as

$$\mathbf{P}(x) = \begin{bmatrix} P_d(x,0) \\ P_d(x,dy) \\ P_d(x,2\,dy) \\ \cdot \\ \cdot \\ \cdot \\ P_d(x,H) \end{bmatrix}, \tag{10}$$

where $dy = H/(M-1)$. The partially discretized model equations can now be written as

$$\frac{d^2}{dx^2}\mathbf{P}(x) + A(x)\mathbf{P}(x) = 0, \quad 0<x<L, \tag{11}$$

where

TABLE I. Comparison of the admittance function parameters used by various investigators. $K(x)$, $R(x)$, and $M(x)$ are the stiffness, damping, and mass of the basilar membrane per unit area, as a function of distance from the stapes.

	$K(x)$ dyn/cm^3	$R(x)$ dyn-s/cm^3	$M(x)$ g/cm^2
Neely (1978)	$10^9 e^{-2x}$	200	0.15
Allen (1977) [a]	$2\times10^9 e^{-3.4x}$	$600e^{-1.7}$	0.10
Lien (1973)	$4.76\times10^8 e^{-1.61}$	$94(x+0.8)^2$	0.20
Peterson and Bogert (1950)	$1.72\times10^9 e^{-2x}$	0	0.143
Siebert (1974)	$2\times10^8 e^{-1.5}$	$5e^{+2.25x}$	0.01

[a] These are the impedance values used to obtain the finite-difference solution shown in Fig. 7. The factor of 2 discrepancy between the values listed here and Allen's (1977) published values has already been noted by Allen and Sondhi (1979).

$$A(x)=\frac{-1}{dy^2}\begin{bmatrix} b(x) & -2 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & & \cdots & & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{bmatrix} \tag{12}$$

is a tridiagonal matrix, and

$$b(x)=2+4i\omega\rho Y(x)\,dy\,. \tag{13}$$

The boundary conditions become:

$$\frac{d}{dx}\,\mathbf{P}(0)=2\omega^2\rho(1\;1\;1\cdots1)^T\,, \tag{14}$$

$$\mathbf{P}(L)=(0\;0\;0\cdots0)^T\,, \tag{15}$$

where the superscript T transposes the row-vectors into column vectors. Note that Eqs. (11)–(13) correspond with Eqs. (1)–(3) and Eqs. (14)–(15) correspond with Eqs. (4)–(5).

The next step is to solve Eqs. (11)–(15) as a set of coupled, second-order differential equations in x. This is done by choosing N discrete points on the x dimension, discretizing Eqs. (11)–(15), and setting up a large $N\times N$ *block-matrix* equation; each element of this block matrix will be an $M\times M$ sub-matrix. This large block matrix is block tridiagonal; moreover, the off-diagonal blocks are all simply scalar multiples of $M\times M$ identity matrices. Because of this special structure, it is relatively easy to solve the block-matrix equation by utilizing a Gaussian block-elimination technique [Isaacson and Keller (1966)]. This procedure was used for all of the numerical results presented in this paper. [1]

III. RESULTS

In this section examples of model solutions are presented. For all solutions the following parameters were used: length of the cochlea $L=3.5$ cm, height of the cochlea $H=0.1$ cm, density of the fluid $\rho=1$ g/cm^3. The pressure difference $P_d(x,y)$ was represented at 246 points in the x dimension and eight points in the y dimension ($M=8$ and $N=246$). Thus the procedure for finding the pressure difference at each point required simultaneous solution of 1960 equations. Some of the computation was performed on a Digital Equipment Corporation PDP-10 computer at the California Institute of Technology and some solutions were obtained on a Texas Instruments 980B computer at Washington University. On the PDP-10 computer a solution for one set of parameters required about 70 s of CPU time.

A. Displacement as a function of distance from the stapes

Displacement of the basilar membrane was obtained as a function of distance from the stapes for ten frequencies: 0.40, 0.57, 0.80, 1.13, 1.60, 2.26, 3.20, 4.52, 6.39, and 9.04 kHz. Figures 2–4 show

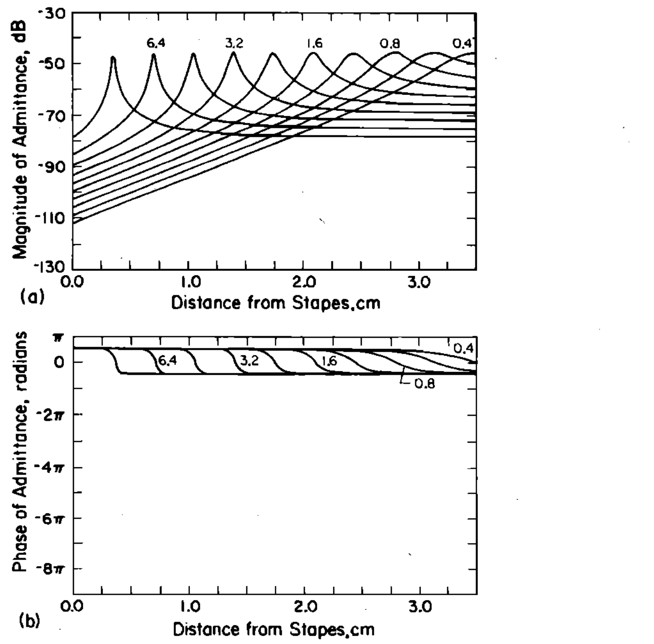

FIG. 2. Basilar membrane admittance as a function of distance from the stapes for ten frequencies, (a) magnitude *re* 1 cm^3/dyn-s and (b) phase. The phase is positive where the stiffness dominates and negative where the mass dominates. The numerals denote frequency in kHz.

magnitude and phase of the admittance of, the pressure difference across, and the displacement of the basilar membrane, with all ten frequencies superimposed. The magnitude is in decibels *re* 1 cgs unit and the phase is in radians. The abscissa represents the x dimension of the model.

The admittance function is computed using the stiffness, damping, and mass values listed on the first line

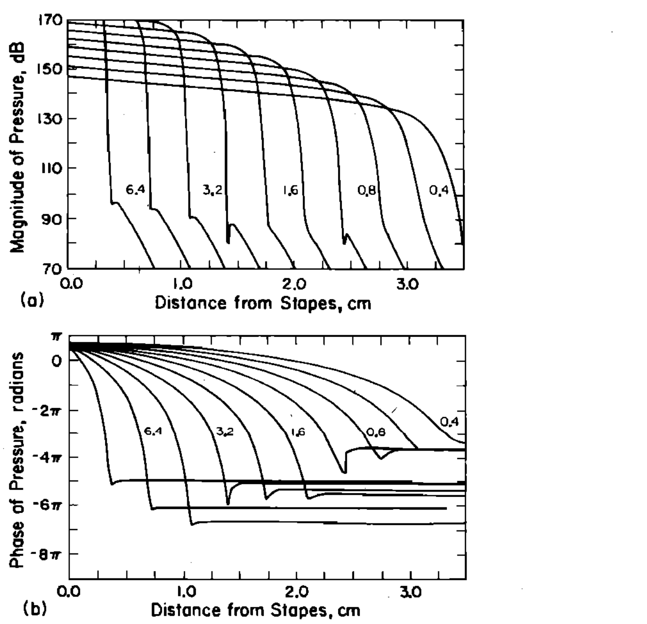

FIG. 3. Pressure difference across the basilar membrane as a function of distance from the stapes for ten frequencies, (a) magnitude *re* 1 dyn/cm^2 and (b) phase. The numerals denote frequency in kHz.

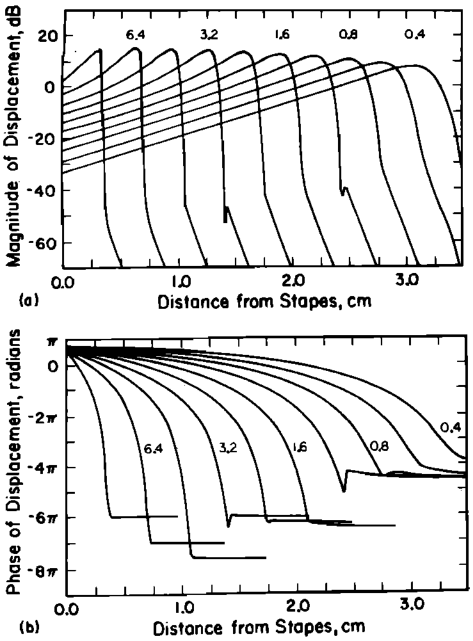

FIG. 4. Basilar membrane displacement as a function of distance from the stapes for ten frequencies, (a) magnitude *re* 1 cm and (b) phase. The numerals denote frequency in kHz.

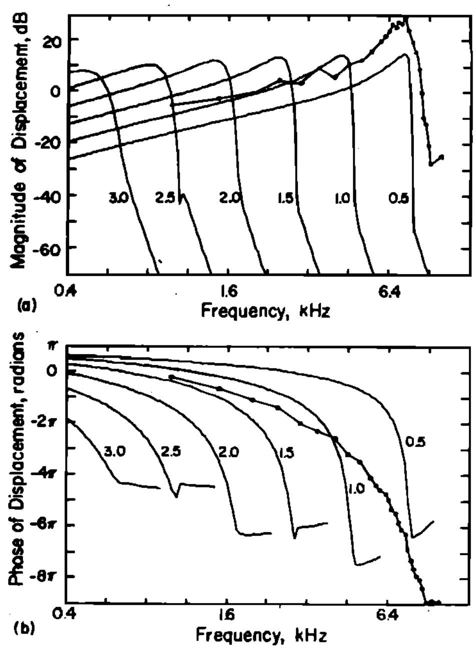

FIG. 5. Basilar membrane displacement as a function of frequency for six positions along the cochlea, (a) magnitude *re* 1 cm and (b) phase. The circles are Rhode's data from animal 69–473 (see text). The model solutions may be interpreted as basilar membrane displacement *re* stapes displacement, whereas Rhode's data show basilar membrane displacement *re* malleus displacement. The numerals denote distance from the stapes in cm.

of Table I. These values were chosen to be similar to the values used by other investigators (also listed in Table I) and to match the human cochlear map [von Békésy (1960)]. The magnitude and phase of the admittance are shown in Fig. 2. The units of admittance are cm^3/dyn-s.

The pressure difference across the basilar membrane is shown in Fig. 3. Note that the pressure difference used in this model is twice the differential pressure used by some investigators. The units of pressure are dyn/cm^2.

The basilar membrane displacement in Fig. 4 is the product of the admittance and the pressure difference divided by the angular frequency, as indicated in Eq. (9). The results shown in Fig. 4 can be interpreted as basilar membrane displacement relative to stapes displacement. The positive sense of stapes displacement and basilar membrane displacement are defined such that where the relative phase of the basilar membrane displacement is zero (or other multiple of 2π) an inward displacement of the stapes coincides with a downward displacement of the basilar membrane. The units of displacement are cm.

B. Displacement as a function of frequency

In Fig. 5 the displacement of the basilar membrane is shown as a function of frequency for six places on the basilar membrane. This required 200 separate solutions of $D(x)$ for 200 different frequencies. For comparison, experimental results of Rhode (1971) from Mössbauer measurements in the cochlea of a living squirrel monkey are superimposed; these data were taken from Zweig *et al.* (1976), Fig. 4.

C. Excitation via scalae walls

Even if the stapes does not move, hearing is possible through bone conduction [Tonndorf, (1976)]. Figure 6 shows the model solution for basilar membrane displacement with the stapes motionless, but with the upper and lower walls vibrating up and down together [the boundary conditions at the stapes and the upper wall, Eqs. (2) and (4), were interchanged]. This condition is intended to simulate hearing by bone conduction. For clarity, only five frequencies are plotted in Fig. 6: 0.4, 0.8, 1.6, 3.2, and 6.4 kHz.

IV. DISCUSSION

A. Admittance and pressure

Figures 2 and 3 were included to show explicitly the component parts (admittance and pressure) of basilar membrane motion. In Fig. 2(b) the phase of the admittance is always $+\pi/2$ at the basal end and $-\pi/2$ at the apical end. The phase indicates the relative contributions of the stiffness and mass components to the total admittance. At the resonant place there two components are balanced and the total phase is zero. The basilar membrane stiffness dominates in the region basal to the resonant place and mass dominates in the region apical to the resonant place. The damping parameter $R(x)$ is a constant so that the relative damping increases from base to apex.

Comparison of the location of the resonant peak in Fig. 2(a) with the location of the maximum displacement in Fig. 4(a) indicates that the maximum displace-

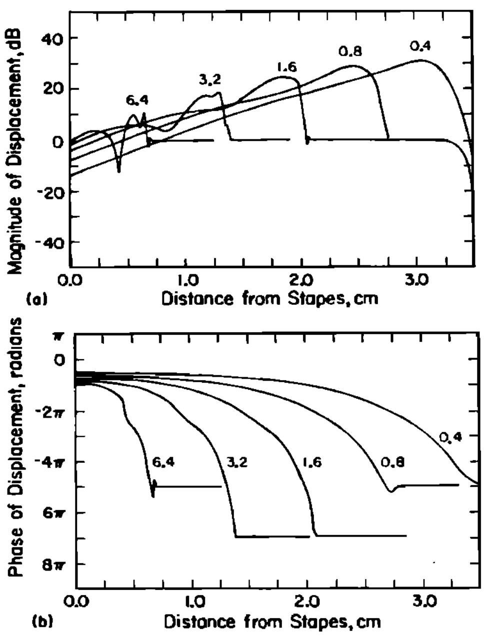

FIG. 6. Basilar membrane displacement as a function of distance from the stapes. (a) magnitude *re* 1 cm and (b) phase. For this figure the stapes was rigid and scala walls were vibrating with unit displacement (1 cm). The numerals denote frequency in kHz.

ment consistently occurs on the basal side of the resonant place. (This is due to the loading of partition by the mass of the fluid.) Thus the admittance is always dominated by its stiffness component at the location of maximum displacement (called the characteristic place).

In Fig. 3(a) the magnitude of the pressure changes relatively little before the characteristic place and very abruptly just beyond that, followed by a break in slope to a more gradual decline in pressure (called the plateau region). Note that the pressure-magnitude curves are always monotonically decreasing up to and slightly beyond the characteristic place.

B. Displacement

The solutions of our model for basilar membrane motion are traveling-wave solutions. Beyond the resonant place the waves are "cut off" and decay exponentially The nature and direction of the traveling wave is not so much determined by the location of the stimulus as it is by the mechanical "tuning" of the basilar membrane. Figure 6 shows the results of solving for basilar membrane displacement when the (normally rigid) upper and lower walls are vibrating instead of the stapes, as in hearing by bone conduction. For the lowest frequency in Fig. 6 the results are nearly the same as for stapes excitation in Fig. 4. As the frequency is increased a standing wave component becomes evident, but the traveling wave dominates and still travels from base to apex. This result appears to be related to von Békésy's (1960) observation of paradoxical waves in the cochlea [see also Siebert, (1974)].

C. Comparison with experimental results

Many hours were spent trying to find model parameters which would produce solutions that fit the experimental data of von Békésy (1960) and Rhode (1971). A simultaneous close fit to both magnitude and phase of Rhode's data was not attainable with any of the many parameter sets attempted, although either could be fit separately with suitable parameters. Allen and Sondhi (1979) and Viergever (1980) have made a similar observation. The parameters selected for this paper were chosen to fit the cochlear map obtained experimentally by von Békésy and compromise on fitting the basilar membrane displacement as observed by Rhode (see Fig. 4). More recent work with a basilar membrane damping function that is negative in a small region basal to the resonant place has produced results which are more typical of Rhode's data. These results are discussed by Kim *et al.* (1980b).

The results of the present model for the low frequencies shown in Fig. 4 agree qualitatively well with the results of von Békésy. The phase drops about 2π radians at the characteristic place for von Békésy's data at 300 Hz and for the model results at 400 Hz. The basilar membrane motion becomes more sharply localized with increasing frequency according to both the model and von Békésy's observations.

Other investigators have modeled the basilar membrane as a beam or plate with significant longitudinal stiffness [Steele (1974), Chadwick *et al.* (1976), and Inselberg and Chadwick (1976)]. If longitudinal stiffness is included in the present model corresponding to the basilar membrane acting as an isotropic plate, the resulting solutions have worse agreement with Rhode's experimental data. The effect of adding a small amount of longitudinal stiffness has been discussed by Allen and Sondhi (1979).

D. Comparison with other solution methods

At the time that finite difference solution method was first demonstrated by this author [Neely (1978)], it was the first complete, direct numerical solution for the two-dimensional cochlear model. Subsequently, it has been used as a standard of comparison for the WKB method of Steele and Taber (1979), the finite-element method of Viergever (1980), and the Green's function method of Matthews (1980).

Prior to the finite difference method, the best available solutions were obtained by Allen's (1977) frequency domain integral-equation method. In Fig. 7 the finite difference solution is compared with the integral-equation solution for the set of parameters attributed to Allen in Table I. The excellent agreement in both absolute magnitude and phase is even better than had been anticipated. This agreement provides support for the validity of the present solution method as well as Allen's method. The only significant difference is in the phase beyond the resonant place (called the phase plateau). The reason for this difference is not known.

The frequency-domain integral-equation method is significantly slower than the finite-difference method

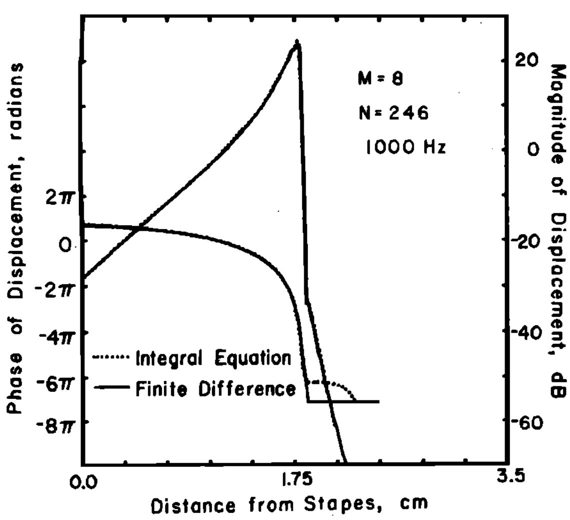

FIG. 7. Comparison of the finite difference solution for basilar membrane displacement (solid line) with the integral equation solution (dotted line) obtained by Allen (1977). In both cases the admittance parameters were those attributed to Allen in Table I.

and requires about twice the amount of computer memory. Sondhi (1978) derived a faster and more compact solution method based on approximating the integral equation by a set of coupled, second-order, ordinary differential equations. Sondhi compares his solution method with Allen's and finds the solutions to be in good agreement.

The finite-element method [Janssen *et al.* (1978); Viergever (1980)] is similar to the finite-difference method in that they both discretize the fluid pressure in two dimensions and can be considered direct numerical solutions. The finite-difference equations retain a closer correspondence to the model equations, which makes analysis somewhat easier. Viergever (personal communication) claims that the finite-element method is more robust and more versatile. The finite-element method more easily allows for local refinement of the discretizing grid. Viergever (1980) takes advantage of this capability in in his frequency-domain solutions by using prior information about local spatial wavelengths in fluid pressure select an appropriate, nonuniform grid. Local refinement of the grid is less useful for time-domain solutions of the cochlear model.

Direct-solution methods are well suited for studying the two-dimensional variation of the fluid pressure, since the two-dimensional pressure is always computed as an intermediate result. Recently, de Boer (1979) has made certain predictions about the pattern of the pressure, velocity, and energy flux which would be expected in the region of the cochlea where waves are "short." Preliminary results with the finite-difference model show agreement with de Boer's predictions in a region close to the basilar membrane and basal to the resonant place.

An important advantage of direct-solution methods is their versatility. Indirect-solution methods, such as the integral-equation method and the WKB method, often place additional constraints on allowable boundary conditions. Several variations of the frequency-domain finite-difference method have been successfully implemented: (1) A two-dimensional cylindrical model, (2) a three-dimensional model, (3) basilar membrane with longitudinal stiffness, (4) compressible fluid, (5) scala-wall excitation, and (6) various combinations of basal and apical boundary conditions.

Allen and Sondhi (1979) have developed a time-domain integral-equation solution method for the two-dimensional model. Matthews (1980) used a very similar time-domain solution method to obtain the steady-state response of the cochlear partition with sinusoidal excitation of the stapes. Matthews finds good agreement between his time-domain results and the finite difference frequency-domain results. The time-domain models are useful in studying certain nonlinear behavior in cochlear mechanics, such as described by Kim and Molnar (1975) and Kim *et al.* (1980a).

E. Computational considerations

With numerical solutions, there is always the question of error introduced due to quantization. Representation of the length of the cochlea (x dimension) by 246 points appears to be adequate for the Neely (1978) parameters for all but the highest frequencies. The 9-kHz solution in Fig. 4 has a noticeable irregularity at the peak-of-the-magnitude. If more points are used to represent the x dimension the 9-kHz solution is smoother, but otherwise looks much the same. Figure 8 compares three 1600-Hz solutions obtained using 100, 200, and 400 points to discretize the x dimension; the results are practically identical. Representation of the scala height (y dimension) by eight points also appears to be adequate. The results shown in Fig. 9 demonstrate that nearly the same solutions are obtained using as many as 16 or as few as four points to represent the y dimension. Apparently, the quantization error has only a minor effect on the solutions presented here.

If computation time and computer memory are limited the finite-difference scheme can be implemented with

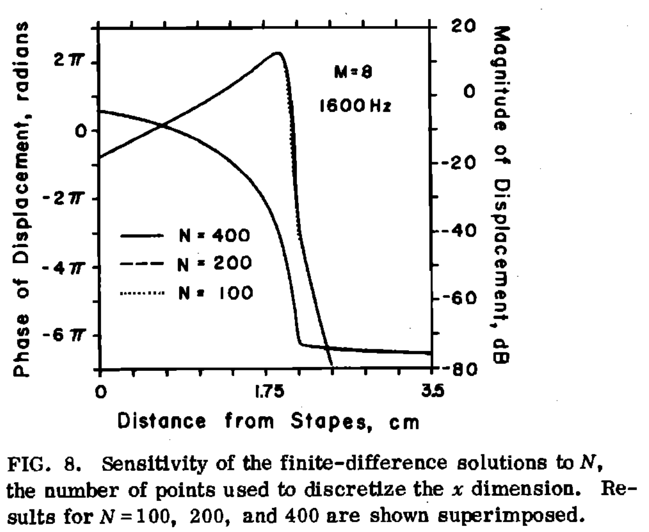

FIG. 8. Sensitivity of the finite-difference solutions to N, the number of points used to discretize the x dimension. Results for $N = 100$, 200, and 400 are shown superimposed.

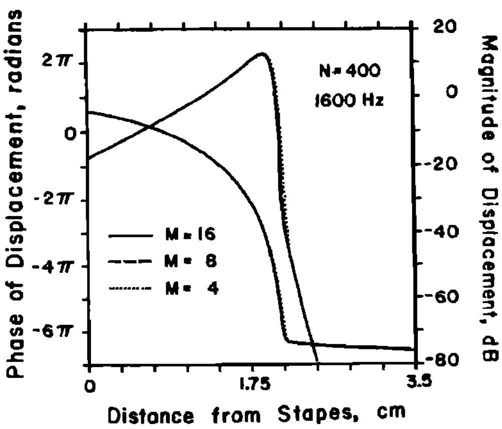

FIG. 9. Sensitivity of the finite difference solutions to M, the number of points used to discretize the y dimension. Results for $M=4$, 8, and 16 are shown superimposed.

as few as three points representing the y dimension. The accuracy can be increased simply by increasing the number of points in the y dimension. This allows one to interact with the model on a short time basis and choose interesting cases for more detailed analysis. Computation time varies approximately with the cube of the number of points in the y dimension (M) and linearly with the number of points in the x dimension (N).

The finite-difference method has also been used as the basis for time-domain solutions of the cochlear model. The key to successful time-domain implementation is that only a small part of the Gaussian block-elimination procedure needs to be repeated each time step. The finite-difference time-domain implementation allows for simultaneous solution of the mechanics of the middle ear, so that interactions of the cochlear partition with eardrum pressure can be studied. Results obtained with the finite-difference time-domain model will be reported at a later time.

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[1]A complete FORTRAN program which implements the numerical procedure and runs on a Texas Instruments 980B minicomputer can be obtained from the author upon request.

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