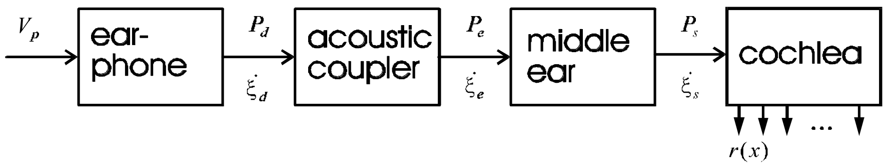
25-Jun-98
STN

Notes on Cochlear Mechanics

1. Peripheral Auditory System

Our objective is to build a model of peripheral auditory mechanics which is sufficient to simulate the neural spike rate of auditory nerve fibers in response to acoustic signals presented to the eardrum. In order to allow for possible presence of otoacoustic emissions in the ear canal, the acoustic stimulus should be specified indirectly. For this reason, we will include an earphone and acoustic coupler in our model. Input to the model will be specified in terms of voltage to the earphone. The four basic stages of the model are (1) earphone, (2) acoustic coupler, (3) middle ear, and (4) cochlea.



Symbols for the physical variables at stage boundaries:

V_p = voltage to the earphone

P_d = pressure to the earphone diaphragm

$\dot{x}_d$ = volume velocity of the earphone diaphragm

P_e = pressure at the eardrum

$\dot{x}_e$ = volume velocity of the middle ear

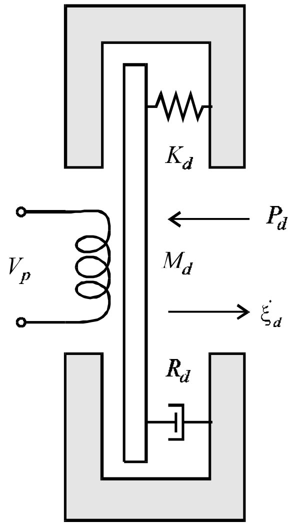
P_s = pressure at the stapes footplate

$\dot{x}_s$ = volume velocity of stapes footplate

$r(x)$ = neural spike discharge rate

We will derive equations (based on physical principles) which will allow us to solve for all of these physical variables, given V_p. We will consider each stage separately and derive appropriate equations of motion. Then we will consider frequency-domain and time-domain solution of these equations.

2. Dynamic Earphone



The model earphone is based on a Beyer DT-48, 200-ohm, dynamic earphone. This model is simple and generic. Other loudspeakers have similar characteristics. The model allows the earphone to have one mechanical degree of freedom (DOF) which is the motion of the diaphragm. This diaphragm has a mass M_d and is attached to the inertial reference frame by a stiffness K_d and a viscous damping R_d. There are two forces acting on the diaphragm. One proportional to the voltage V_p and the other proportional to the pressure P_d.

$$M_d \ddot{x}_d + (R_d + R_p)\dot{x}_d + K_d x_d = (C_p V_p - A_d P_d) A_d$$

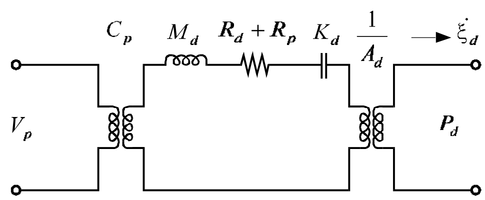
is the equation of motion for the earphone. The additional damping term R_p has been added to account for the energy lost due to induced electrical currents as a result of motion of the voice coil through a magnetic field. The voice coil parameters are defined as

$$C_p = \frac{B\ell}{10 R_v}$$

$$R_p = 10^{-9} \frac{(B\ell)^2}{R_v}$$

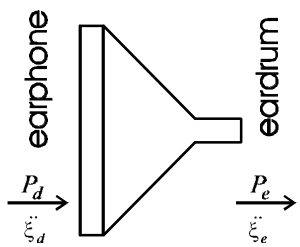
where B the magnetic flux density, ℓ is the length of the wire in the voice coil winding, and R_v is the electrical resistance of the voice coil.

We can represent the earphone by an electrical analog circuit like this.



The multipliers above the transformers in this diagram indicate the voltage gain going from left to right.

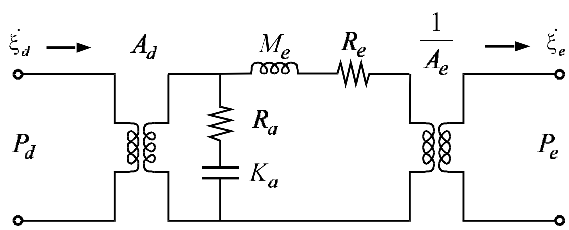
3. Acoustic Coupler



The model of the acoustic coupling between the eardrum and the eardrum is based on a funnel-shaped device used in laboratory experiments. The air in the small tube on one end of the coupler is treated as a lumped mass. A force is extertED on this mass by the pressure at the earphone P_d and the pressure at the eardrum P_e. The air in the conical cavity is treated as a spring K_a which couples the air in the tube with the diaphragm of the eardrum. Damping is included to represent losses in the conical cavity and the tube, R_a and R_e. The equation of motion becomes

$$M_e \ddot{x}_e + R_e \dot{x}_e = (A_d P_d - A_a P_e) A_e$$

An electrical analog circuit looks like this.



We can combine the equations for earphone and acoustic coupler to eliminate P_d

$$\frac{M_d}{A_d^2}\ddot{x}_d + \left(\frac{R_d + R_p + R_a}{A_d^2}\right)\dot{x}_d + \left(\frac{K_d + K_a}{A_d^2}\right)x_d - \left(\frac{R_a}{A_e A_d}\right)\dot{x}_e - \left(\frac{K_a}{A_e A_d}\right)x_e = \frac{C_p}{A_d}V_p$$

$$\frac{M_e}{A_e^2}\ddot{x}_e + \left(\frac{R_e + R_a}{A_e^2}\right)\dot{x}_e + \left(\frac{K_a}{A_e^2}\right)x_e - \left(\frac{R_a}{A_e A_d}\right)\dot{x}_d - \left(\frac{K_a}{A_e A_d}\right)x_d = -P_e$$

We can check the earphone model independent of the influence of the middle ear and cochlea by setting x_e to zero and solving for x_d given V_p.

$$\frac{M_d}{A_d^2}\ddot{x}_d + \left(\frac{R_d + R_p + R_a}{A_d^2}\right)\dot{x}_d + \left(\frac{K_d + K_a}{A_d^2}\right)x_d = \frac{C_p}{A_d}V_p$$

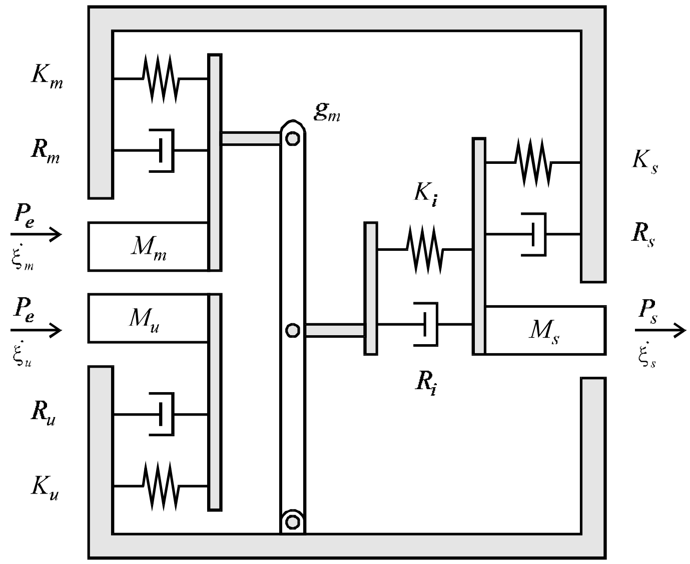
$$\left(\frac{R_a}{A_e A_d}\right)\dot{x}_d + \left(\frac{K_a}{A_e A_d}\right)x_d = P_e$$

Solutions of these equation can be compared with responses recorded from a microphone covering the small end of the acoustic coupler. In the frequency domain

$$P_e = \frac{C_p}{A_e}\left[\frac{R_a + K_a \frac{1}{iw}}{M_d iw + (R_d + R_p + R_a) + (K_d + K_a)\frac{1}{iw}}\right]V_p$$

when x_e is zero.

4. Middle Ear



The middle ear is allowed to have three degrees of freedom (1) effective motion of the eardrum and malleus, (2) ineffective or unused motion of the eardrum, and (3) motion of the stapes. The incus is assumed to move with the stapes. The first equation of motion represents the unused mode of the middle ear.

$$M_u\ddot{x}_u + R_u\dot{x}_u + K_u x_u = (A_u P_e)A_u$$

The next equation represents the malleus.

$$M_m\ddot{x}_m + R_m\dot{x}_m + K_m x_m = (A_m P_e - g_m F_i)A_m$$

$$F_i = R_i\left(g_m\frac{\dot{x}_m}{A_m} - \frac{\dot{x}_s}{A_s}\right) + K_i\left(g_m\frac{x_m}{A_m} - \frac{x_s}{A_s}\right)$$

The pressure at the stapes pushes to the left.

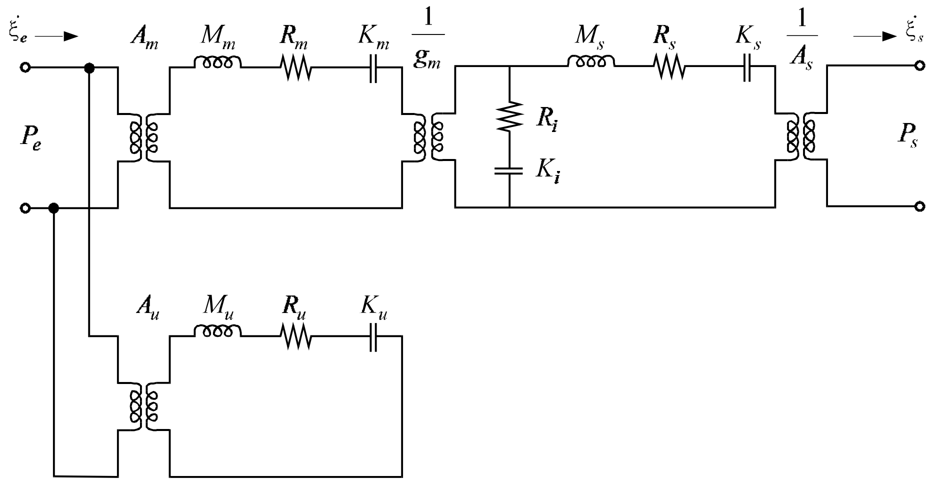
$$M_s\ddot{x}_s + R_s\dot{x}_s + K_s x_s = (F_i - A_s P_s)A_s$$

We also need a continuity equation to complete the description of the middle-ear model.

$$x_e = x_m + x_u$$

The continuity equation indicates that the volume velocity at the output of the acoustic coupler is equal to the sum of the volume velocities of the malleus and unused modes of the eardrum.

An electrical analog of the middle-ear model looks like this



We can combine the middle-ear equations above to obtain three equations of motion.

$$\left(\frac{M_u}{A_u^2}\right)\ddot{x}_u + \left(\frac{R_u}{A_u^2}\right)\dot{x}_u + \left(\frac{K_u}{A_u^2}\right)x_u = P_e$$

$$\left(\frac{M_m}{A_m^2}\right)\ddot{x}_m + \left(\frac{R_m + g_m^2 R_i}{A_m^2}\right)\dot{x}_m + \left(\frac{K_m + g_m^2 K_i}{A_m^2}\right)x_m - \left(\frac{g_m R_i}{A_m A_s}\right)\dot{x}_s - \left(\frac{g_m K_i}{A_m A_s}\right)x_s = P_e$$

$$\left(\frac{M_s}{A_s^2}\right)\ddot{x}_s + \left(\frac{R_s + R_i}{A_s^2}\right)\dot{x}_s + \left(\frac{K_s + K_i}{A_s^2}\right)x_s - \left(\frac{g_m R_i}{A_m A_s}\right)\dot{x}_m - \left(\frac{g_m K_i}{A_m A_s}\right)x_m = -P_s$$

5. Combining, Earphone, Acoustic Coupler, and Middle Ear

We can combine the equations for the earphone, middle ear and acoustic coupler into four equations of motion:

$$\frac{M_d}{A_d^2}\ddot{x}_d + \left(\frac{R_d + R_p + R_a}{A_d^2}\right)\dot{x}_d + \left(\frac{K_d + K_a}{A_d^2}\right)x_d - \frac{R_a}{A_e A_d}\dot{x}_e - \frac{K_a}{A_e A_d}x_e = \frac{C_p}{A_d}V_p$$

$$\left(\frac{M_e}{A_e^2} + \frac{M_u}{A_u^2}\right)\ddot{x}_e + \left(\frac{R_a + R_e}{A_e^2} + \frac{R_u}{A_u^2}\right)\dot{x}_e + \left(\frac{K_a}{A_e^2} + \frac{K_u}{A_u^2}\right)x_e - \frac{R_a}{A_e A_d}\dot{x}_d - \frac{K_a}{A_e A_d}x_d - \frac{M_u}{A_u^2}\ddot{x}_m - \frac{R_u}{A_u^2}\dot{x}_m - \frac{K_u}{A_u^2}x_m = 0$$

$$\left(\frac{M_m}{A_m^2} + \frac{M_u}{A_u^2}\right)\ddot{x}_m + \left(\frac{R_m + g_m^2 R_i}{A_m^2} + \frac{R_u}{A_u^2}\right)\dot{x}_m + \left(\frac{K_m + g_m^2 K_i}{A_m^2} + \frac{K_u}{A_u^2}\right)x_m - \frac{g_m R_i}{A_m A_s}\dot{x}_s - \frac{g_m K_i}{A_m A_s}x_s - \frac{M_u}{A_u^2}\ddot{x}_e - \frac{R_u}{A_u^2}\dot{x}_e - \frac{K_u}{A_u^2}x_e = 0$$

$$\frac{M_s}{A_s^2}\ddot{x}_s + \frac{R_s + R_i}{A_s^2}\dot{x}_s + \frac{K_s + K_i}{A_s^2}x_s - \frac{g_m R_i}{A_m A_s}\dot{x}_m - \frac{g_m K_i}{A_m A_s}x_m = -P_s$$

where $x_u = x_e - x_m$.

We can rewrite these equations of motion in the terms of a state vector $\underline{x}$ and matrices of mechanical parameters such that

$$[M]\ddot{\underline{x}} + [R]\dot{\underline{x}} + [K]\underline{x} = \underline{P}$$

where

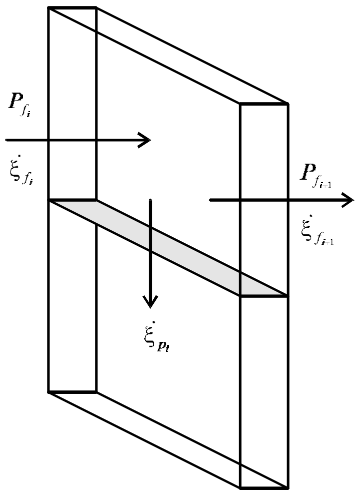
$$\underline{x} = \begin{bmatrix} x_d \\ x_e \\ x_m \\ x_s \end{bmatrix}, \qquad \underline{P} = \begin{bmatrix} \frac{C_p}{A_d}V_p \\ 0 \\ 0 \\ -P_s \end{bmatrix},$$

$$[M] = \begin{bmatrix} \frac{M_d}{A_d^2} & 0 & 0 & 0 \\ 0 & \frac{M_e}{A_e^2} + \frac{M_u}{A_u^2} & -\frac{M_u}{A_u^2} & 0 \\ 0 & -\frac{M_u}{A_u^2} & \frac{M_m}{A_m^2} + \frac{M_u}{A_u^2} & 0 \\ 0 & 0 & 0 & \frac{M_s}{A_s^2} \end{bmatrix}, \qquad [R] = \begin{bmatrix} \frac{R_d + R_p + R_a}{A_d^2} & -\frac{R_a}{A_a A_d} & 0 & 0 \\ -\frac{R_a}{A_e A_d} & \frac{R_a + R_e}{A_e^2} + \frac{R_u}{A_u^2} & -\frac{R_u}{A_u^2} & 0 \\ 0 & -\frac{R_u}{A_u^2} & \frac{R_m + g_m^2 R_i}{A_m^2} + \frac{R_u}{A_u^2} & -\frac{g_m R_i}{A_m A_s} \\ 0 & 0 & -\frac{g_m R_i}{A_m A_s} & \frac{R_s + R_i}{A_s^2} \end{bmatrix},$$

and

$$[K] = \begin{bmatrix} \frac{K_d + K_a}{A_d^2} & -\frac{K_a}{A_a A_d} & 0 & 0 \\ -\frac{K_a}{A_a A_d} & \frac{K_a}{A_a^2} + \frac{K_u}{A_u^2} & -\frac{K_u}{A_u^2} & 0 \\ 0 & -\frac{K_u}{A_u^2} & \frac{K_m + g_m^2 K_i}{A_m^2} + \frac{K_u}{A_u^2} & -\frac{g_m K_i}{A_m A_s} \\ 0 & 0 & -\frac{g_m K_i}{A_m A_s} & \frac{K_s + K_i}{A_s^2} \end{bmatrix}.$$

6. Cochlear macromechanics



We need to derive the equations of motion for the fluid in the cochlear chambers. We will only represent the longitudinal spatial dimension and treat the cochlea as a series of radial cross-sections (RCS). The fluid is assumed to be incompressible so the volume of fluid leaving the upper chamber of an RCS must be equal to the fluid entering the upper chamber minus the volume displacement of the cochlear partition.

$$x_{f_{i+1}} = x_{f_i} - x_{p_i}$$

The motion of the fluid in the lower chamber is assumed to be equal and opposite to the fluid in the upper chamber due to the connection of the fluid chambers at the helicotrema and the incompressibility of the fluid.

If the cross-sectional area of each chamber is A_{c_i} and the thickness of the RCS is dx, then the mass of the fluid (including both chambers) will be

$$M_{f_i} = 2rA_{c_i}dx$$

where r is the density of the fluid.

The equations of motion for the RCS are

$$\frac{M_{f_i}}{A_{c_i}^2}\ddot{x}_{f_{i+1}} + \frac{R_{f_i}}{A_{c_i}^2}\dot{x}_{f_{i+1}} = P_{f_i} - P_{f_{i+1}}$$

$$\frac{M_{p_i}}{A_{p_i}^2}\ddot{x}_{p_i} + S_{p_i} = P_{f_i}$$

where R_{f_i} represents viscous damping due to longitudinal fluid motion, M_{p_i} is the mass of the partition and S_{p_i} represents other forces on the partition due to cochlear micromechanics. These other forces will depend on partition velocity and displacement, but not on partition acceleration.

We can use the continuity equation (conservation of fluid volume) to combine the two equations of motion into a single equation

$$\frac{M_{f_{i-1}}}{A_{c_{i-1}}^2}\ddot{x}_{f_i} + \frac{R_{f_{i-1}}}{A_{c_{i-1}}^2}\dot{x}_{f_i} = \left[\frac{M_{p_{i-1}}}{A_{p_{i-1}}^2}\left(\ddot{x}_{f_{i-1}} - \ddot{x}_{f_i}\right) + S_{p_{i-1}}\right] - \left[\frac{M_{p_i}}{A_{p_i}^2}\left(\ddot{x}_{f_i} - \ddot{x}_{f_{i+1}}\right) + S_{p_i}\right]$$

or

$$\left(\frac{M_{f_{i-1}}}{A_{c_{i-1}}^2} + \frac{M_{p_{i-1}}}{A_{p_{i-1}}^2} + \frac{M_{p_i}}{A_{p_i}^2}\right)\ddot{x}_{f_i} - \frac{M_{p_{i-1}}}{A_{p_{i-1}}^2}\ddot{x}_{f_{i-1}} - \frac{M_{p_i}}{A_{p_i}^2}\ddot{x}_{f_{i+1}} + \frac{R_{f_{i-1}}}{A_{c_{i-1}}^2}\dot{x}_{f_i} = S_{p_{i-1}} - S_{p_i}$$

This equation can be used in a time-domain model to solve for cochlear macromechanics since the right-hand side will be known at each time-step if the velocity and displacement of the partition are known.

There are variations in the equation of motion at the stapes and helicotrema due to boundary conditions. If there are N sections numbered $i = 0,1,\dots,N-1$, then the stapes boundary is at $i = 0$ and the helicotrema boundary is at $i = N-1$. At the stapes we will assume that $P_{f_0} = P_s$ and $x_{f_0} = x_s$. At the helicotrema we have $x_{f_N} = 0$ and $S_{p_{N-1}} = \dfrac{R_h}{A_{P_{N-1}}^2}$.

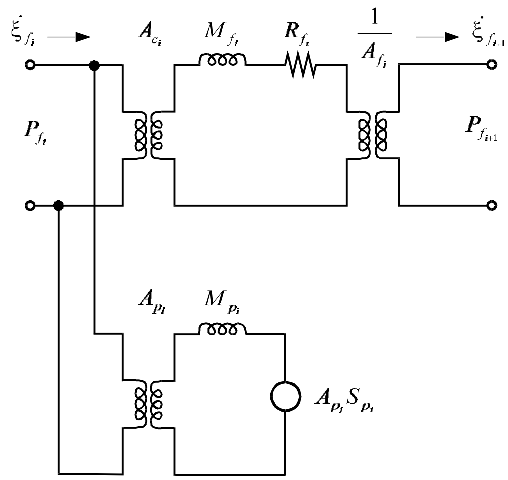
Combining these boundary conditions with the equations of motion, we obtain

$$\frac{M_{p_0}}{A_{p_0}^2}\left[\ddot{x}_s - \ddot{x}_{f_1}\right] + S_{p_0} = P_s$$

and

$$\left(\frac{M_{f_{N-2}}}{A_{c_{N-2}}^2} + \frac{M_{p_{N-2}}}{A_{p_{N-2}}^2}\right)\ddot{x}_{f_{N-1}} - \frac{M_{p_{N-2}}}{A_{p_{N-2}}^2}\ddot{x}_{f_{N-2}} + \frac{R_{f_{N-2}}}{A_{c_{N-2}}^2}\dot{x}_{f_{N-2}} = S_{p_{N-2}} - \frac{R_h}{A_{p_{N-2}}^2}$$

An electrical analog circuit for a single RCS looks like this



where S_{p_i} depends on the micromechanics of the cochlear partition.

7. Cochlear micromechanics (1-DOF)

We will with a simple representation of cochlear micromechanics with only 1 degree of freedom (DOF) in each radial cross section (RCS), namely the displacement of the basilar membrane.

$$x_{b_i} = -\frac{x_{p_i}}{A_{p_i}}$$

The equation of motion for the basilar membrane will be

$$M_{b_i}\ddot{x}_{b_i} + R_{b_i}\dot{x}_{b_i} + K_{b_i}x_{b_i} = -A_{p_i}P_{f_i}$$

or

$$\frac{M_{b_i}}{A_{p_i}^2}\ddot{x}_{p_i} + \frac{R_{b_i}}{A_{p_i}^2}\dot{x}_{p_i} + \frac{K_{b_i}}{A_{p_i}^2}x_{p_i} = P_{f_i}$$

This equation has the from of the partition equation used in the previous section

with

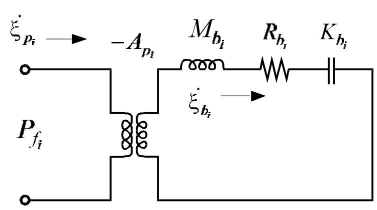
$$M_{p_i} = M_{b_i}$$

and (for 1-DOF)

$$S_{p_i} = \frac{R_{b_i}}{A_{p_i}^2}\dot{x}_{p_i} + \frac{K_{b_i}}{A_{p_i}^2}x_{p_i}$$

This representation of cochlear micromechanics is sufficient to complete our model of the cochlea and to simulate traveling waves on the basilar membrane. We need more than 1 DOF, however, to simulate neural-like tuning.

An electrical analog circuit looks like this.



8. Time-domain solution

Before we add more degrees of freedom to cochlear micromechanics, we will consider how to combine the equations for the cochlea (macromechanics and micromechanics) with the equations for the middle ear, acoustic coupler and earphone to obtain time-domain and frequency-domain numerical solutions.

First we need to combine the stapes boundary conditions for cochlea and middle ear

$$\frac{M_{p_0}}{A_{p_0}^2}\left[\ddot{x}_s - \ddot{x}_{f_1}\right] + S_{p_0} = -\left[\left(\frac{M_s}{A_s^2}\right)\ddot{x}_s + \left(\frac{R_s + R_i}{A_s^2}\right)\dot{x}_s + \left(\frac{K_s + K_i}{A_s^2}\right)x_s - \left(\frac{g_m R_i}{A_m A_s}\right)\dot{x}_m - \left(\frac{g_m K_i}{A_m A_s}\right)x_m\right]$$

or

$$\left(\frac{M_s}{A_s^2} + \frac{M_{p_0}}{A_{p_0}^2}\right)\ddot{x}_s - \left(\frac{M_{p_0}}{A_{p_0}^2}\right)\ddot{x}_{f_1} + S_{p_0} + \left(\frac{R_s + R_i}{A_s^2}\right)\dot{x}_s + \left(\frac{K_s + K_i}{A_s^2}\right)x_s - \left(\frac{g_m R_i}{A_m A_s}\right)\dot{x}_m - \left(\frac{g_m K_i}{A_m A_s}\right)x_m = 0$$

We can now rewrite the entire set of equations of motion in the terms of a state vector $\underline{x}$ and matrices of mechanical parameters such that

$$[M]\ddot{\underline{x}} + \underline{S} = \underline{P}$$

where

$$\underline{x} = \begin{bmatrix} x_d \\ x_e \\ x_m \\ x_s \\ x_{f_1} \\ x_{f_2} \\ \vdots \\ x_{f_{N-2}} \\ x_{f_{N-1}} \end{bmatrix} \qquad \underline{P} = \begin{bmatrix} \frac{C_p}{A_d}V_p \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$[M] = \begin{bmatrix} m_d & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_e & -m_u & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -m_u & m_m & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & m_s + m_{p_0} & -m_{p_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -m_{p_0} & m_{f_1} & -m_{p_1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_{p_1} & m_{f_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & m_{f_{N-2}} & -m_{p_{N-2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -m_{p_{N-2}} & m_{f_{N-1}} \end{bmatrix}$$

where $m_d = \frac{M_d}{A_d^2}$, $m_e = \left(\frac{M_e}{A_e^2} + \frac{M_u}{A_u^2}\right)$, $m_u = \frac{M_u}{A_u^2}$, $m_m = \left(\frac{M_m}{A_m^2} + \frac{M_u}{A_u^2}\right)$, $m_s = \frac{M_s}{A_s^2}$, $m_{p_i} = \frac{M_{p_i}}{A_{p_i}^2}$, $m_{f_i} = \left(\frac{M_{f_{i-1}}}{A_{c_{i-1}}^2} + \frac{M_{p_{i-1}}}{A_{p_{i-1}}^2} + \frac{M_{p_i}}{A_{p_i}^2}\right)$. The mass matrix must be inverted at each time step for a time-domain solution. This inversion is made easier by the fact that the matrix is tridiagonal.

Finally,

$$\underline{S} = \begin{bmatrix} \frac{R_d + R_p + R_a}{A_d^2}\dot{x}_d + \frac{K_d + K_a}{A_d^2}x_d - \frac{R_a}{A_e A_d}\dot{x}_e - \frac{K_a}{A_e A_d}x_e \\ \left(\frac{R_a + R_e}{A_e^2} + \frac{R_u}{A_u^2}\right)\dot{x}_e + \left(\frac{K_a}{A_e^2} + \frac{K_u}{A_u^2}\right)x_e - \frac{R_a}{A_e A_d}\dot{x}_d - \frac{K_a}{A_e A_d}x_d - \frac{R_u}{A_u^2}\dot{x}_m + \frac{K_u}{A_u^2}x_m \\ \left(\frac{R_m + g_m^2 R_i}{A_m^2} + \frac{R_u}{A_u^2}\right)\dot{x}_m + \left(\frac{K_m + g_m^2 K_i}{A_m^2} + \frac{K_u}{A_u^2}\right)x_m - \frac{R_u}{A_u^2}\dot{x}_e - \frac{K_u}{A_u^2}x_e - \frac{g_m R_i}{A_m A_s}\dot{x}_s - \frac{g_m K_i}{A_m A_s}x_s \\ S_{p_0} + \frac{R_s + R_i}{A_s^2}\dot{x}_s + \frac{K_s + K_i}{A_s^2}x_s - \frac{g_m R_i}{A_m A_s}\dot{x}_m - \frac{g_m K_i}{A_m A_s}x_m \\ \frac{R_{f_0}}{A_{c_0}^2}\dot{x}_{f_1} + S_{p_1} - S_{p_0} \\ \frac{R_{f_1}}{A_{c_1}^2}\dot{x}_{f_2} + S_{p_2} - S_{p_1} \\ \vdots \\ \frac{R_{f_{N-3}}}{A_{c_{N-3}}^2}\dot{x}_{f_{N-2}} + S_{p_{N-2}} - S_{p_{N-3}} \\ \frac{R_{f_{N-2}}}{A_{c_{N-2}}^2}\dot{x}_{f_{N-1}} + \frac{R_h}{A_{p_{N-1}}^2} - S_{p_{N-1}} \end{bmatrix}$$

Note that this vector depends only on velocities and displacements of the state variables.

We can proceed with a time-domain solution of these equations by assuming that at some initial time t_0 we know that "state" of the system in terms of the velocity vector $\underline{\dot{x}}(t_0)$ and the displacement vector $\underline{x}(t_0)$ so that we can construct the influence vector $\underline{S}(t_0)$. We also assume that we know the value of any external forces applied so we can construct the vector $\underline{P}(t_0)$. In order to solve for the acceleration $\underline{\ddot{x}}(t_0)$, we need the matrix equation

$$[M]\underline{\ddot{x}}(t_0) = \underline{P}(t_0) - \underline{S}(t_0)$$

Now we need to extrapolate what we know about the state variables at t_0 in order to estimate the velocity vector $\underline{\dot{x}}(t_1)$ and the displacement vector $\underline{x}(t_1)$ at the next time t_1. We start by assuming that the acceleration will be approximately constant between t_0 and t_1 so that our best estimate of $\underline{\dot{x}}(t_1)$ will be

$$\underline{\dot{x}'}(t_1) = \underline{\dot{x}}(t_0) + (t_1 - t_0)\underline{\ddot{x}}(t_0).$$

Having an estimate of the velocity at t_1, our best estimate of the displacement at t_1 will be

$$\underline{x'}(t_1) = \underline{x}(t_0) + \frac{1}{2}(t_1 - t_0)\left(\underline{\dot{x}}(t_0) + \underline{\dot{x}'}(t_1)\right)$$

We can improve our initial predictions of velocity $\underline{\dot{x}}(t_1)$ and displacement $\underline{x}(t_1)$ by solving for the acceleration at time t_1 and repeating the integration steps.

$$[M]\ddot{\underline{x}}'(t_1) = \underline{P}(t_1) - \underline{S}(t_1)$$

$$\dot{\underline{x}}''(t_1) = \dot{\underline{x}}(t_0) + \frac{1}{2}(t_1 - t_0)\left(\ddot{\underline{x}}(t_0) + \ddot{\underline{x}}'(t_1)\right)$$

$$\underline{x}''(t_1) = \underline{x}(t_0) + \frac{1}{2}(t_1 - t_0)\left(\dot{\underline{x}}(t_0) + \dot{\underline{x}}''(t_1)\right)$$

These three steps can be repeated, if necessary, to obtain further improvements in the estimates of $\dot{\underline{x}}(t_1)$ and $\underline{x}(t_1)$. In practice, it seems sufficient to use the values obtained after a single improvement.

$$\dot{\underline{x}}(t_1) = \dot{\underline{x}}''(t_1)$$

$$\underline{x}(t_1) = \underline{x}''(t_1)$$

Now the velocity and displacement are known at time t_1 and the same procedure is repeated to obtain the state variables at succeeding time steps.

9. Frequency-domain solution

To obtain frequency-domain solutions we assume harmonic time dependence e^{iwt}. We can rewrite the equations of motion as a complex matrix equation

$$[Z]\underline{\dot{x}} = \underline{P}$$

where

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_d \\ \dot{x}_e \\ \dot{x}_m \\ \dot{x}_s \\ \dot{x}_{f_1} \\ \dot{x}_{f_2} \\ \vdots \\ \dot{x}_{f_{N-2}} \\ \dot{x}_{f_{N-1}} \end{bmatrix} \qquad \underline{P} = \begin{bmatrix} \frac{C_p}{A_d} V_p \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$[Z] = \begin{bmatrix} z_d & -z_a & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -z_a & z_e & -z_u & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -z_u & z_m & -z_i & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -z_i & z_s + z_{p_0} & -z_{p_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -z_{p_0} & z_{f_1} & -z_{p_1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -z_{p_1} & z_{f_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & z_{f_{N-2}} & -z_{p_{N-2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -z_{p_{N-2}} & z_{f_{N-1}} \end{bmatrix}$$

where

$$z_d = \frac{M_d}{A_d^2} iw + \frac{R_d + R_p + R_a}{A_d^2} + \frac{K_d + K_a}{A_d^2} \frac{1}{iw}$$

$$z_a = \frac{R_a}{A_e A_d} + \frac{K_a}{A_e A_d} \frac{1}{iw}$$

$$z_e = \left(\frac{M_e}{A_e^2} + \frac{M_u}{A_u^2} \right) iw + \left(\frac{R_a + R_e}{A_e^2} + \frac{R_u}{A_u^2} \right) + \left(\frac{K_a}{A_e^2} + \frac{K_u}{A_u^2} \right) \frac{1}{iw}$$

$$z_u = \frac{R_u}{A_u^2} + \frac{K_u}{A_u^2} \frac{1}{iw}$$

$$z_m = \left(\frac{M_m}{A_m^2} + \frac{M_u}{A_u^2} \right) iw + \left(\frac{R_m + g_m^2 R_i}{A_m^2} + \frac{R_u}{A_u^2} \right) + \left(\frac{K_m + g_m^2 K_i}{A_m^2} + \frac{K_u}{A_u^2} \right) \frac{1}{iw}$$

$$z_i = \frac{g_m R_i}{A_m A_s} + \frac{g_m K_i}{A_m A_s} \frac{1}{iw}$$

$$z_s = \left(\frac{M_s}{A_s^2} \right) iw + \left(\frac{R_s + R_i}{A_s^2} \right) + \left(\frac{K_s + K_i}{A_s^2} \right) \frac{1}{iw}$$

$$z_{p_i} = Z_{p_i}$$

$$z_{f_i} = \frac{M_{f_{i-1}}}{A_{c_{i-1}}^2} iw + \frac{R_{f_{i-1}}}{A_{c_{i-1}}^2} + Z_{p_i} + Z_{p_{i-1}}$$

The impedance matrix $[Z]$ must be inverted at each time step for a time-domain solution. This inversion is made easier by the fact that the matrix is tridiagonal.

The partition impedance Z_{p_i} characterizes the cochlear micromechanics and is defined as the ratio of the pressure and velocity.

$$Z_{p_i} = \frac{P_{f_i}}{\dot{x}_{p_i}} = \frac{P_{f_i}}{\dot{x}_{f_i} - \dot{x}_{f_{i+1}}}$$

For micromechanics with 1-DOF

$$Z_{p_i} = M_{p_i} iw + R_{p_i} + K_{p_i} \frac{1}{iw}$$

10. Resonant Tectorial Membrane (2-DOF)

We need 2-DOF to represent the tectorial membrane as a separate mechanical element. We will assume that the displacement of the hair bundle of the outer hair cells x_c is a linear combination of basilar membrane displacement x_b and tectorial membrane displacement x_t.

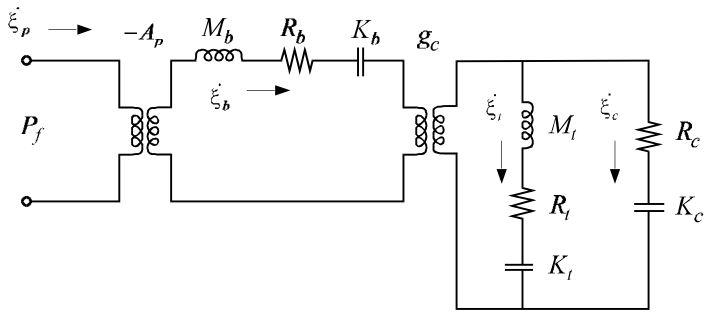
$$x_c = g_c x_b - x_t$$

where g_c is the shear lever-gain between upward displacement of the basilar membrane and radial displacement of the reticular lamina. The equations of motion are

$$M_b \ddot{x}_b + R_b \dot{x}_b + K_b x_b + g_c R_c \dot{x}_c + g_c K_c x_c = -A_p P_f$$

$$M_t \ddot{x}_t + R_t \dot{x}_t + K_t x_t - R_c \dot{x}_c - K_c x_c = 0$$

An electrical analog circuit looks like this.



For time domain solutions with 2-DOF micromechanics we have

$$S_p = \frac{-1}{A_p}\left[R_b \dot{x}_b + K_b x_b + g_c R_c \dot{x}_c + g_c K_c x_c\right]$$

and, in order to integrate x_t

$$\ddot{x}_t = \frac{-1}{M_t}\left[R_t \dot{x}_t + K_t x_t - R_c \dot{x}_c - K_c x_c\right]$$

For frequency-domain solutions we will have

$$Z_p = Z_b + g_c H_c Z_c$$

where

$$Z_b = M_b iw + R_b + K_b \frac{1}{iw}$$

$$Z_t = M_t iw + R_t + K_t \frac{1}{iw}$$

$$Z_c = R_c + K_c \frac{1}{iw}$$

$$H_c = \frac{g_c Z_t}{Z_t + Z_c}$$

11. Outer Hair Cell Motility (3-DOF)

In order to incorporate outer hair cell length changes into the micromechanics, we need to add another mechanical degree of freedom. We will allow the reticular lamina to have a separate motion so that the relative distance between the basilar membrane and the reticular lamina will change when the outer hair cell length changes. In addition, we will assume that the outer hair cells exert a force F_o between the basilar membrane and the reticular lamina that is proportional to the voltage across the cell membrane. The equations of motion for the 3-DOF micromechanics will be

$$M_b\ddot{x}_b + R_b\dot{x}_b + K_b x_b + g_c R_c \dot{x}_c + g_c K_c x_c = -A_p P_f$$

$$M_t\ddot{x}_t + R_t\dot{x}_t + K_t x_t - R_c \dot{x}_c - K_c x_c = 0$$

$$M_r\ddot{x}_r + R_r\dot{x}_r + K_r x_r - g_c R_c \dot{x}_c - g_c K_c x_c - F_o = 0$$

where

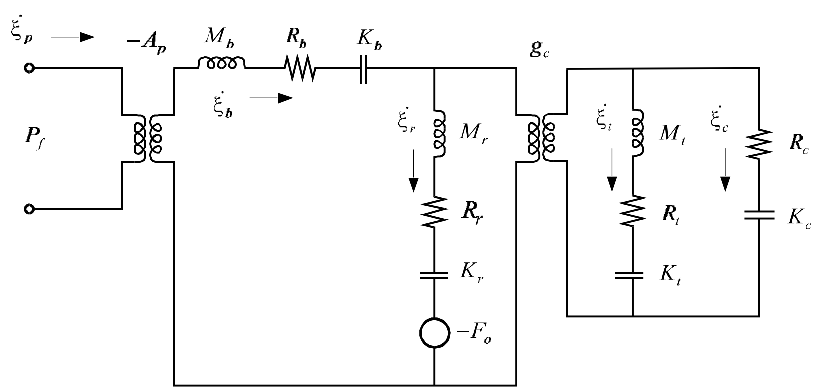
$$x_c = g_c(x_b - x_r) - x_t$$

$$F_o = g_r K_o V_m$$

and V_m is the voltage across the outer hair cell membrane. This voltage is controlled by the displacement of the hair bundle at the top of the cell and dependent on the conductance G_m and capacitance C_m of the hair cell.

$$C_m\dot{V}_m + G_m V_m = g_f G_m x_o - g_r K_o \dot{x}_r$$

An electrical analog circuit looks like this



For time-domain solutions we will have

$$S_p = \frac{-1}{A_p}\left[R_b\dot{x}_b + K_b x_b + g_c R_c \dot{x}_c + g_c K_c x_c\right]$$

with

$$\ddot{x}_t = \frac{-1}{M_t}\left[R_t\dot{x}_t + K_t x_t - g_c R_c \dot{x}_c - g_c K_c x_c\right]$$

$$\ddot{x}_r = \frac{-1}{M_r}\left[R_r\dot{x}_r + K_r x_r - R_c \dot{x}_c - K_c x_c - g_r K_o V_m\right]$$

$$\dot{V}_m = \frac{g_f G_m}{C_m} x_c - \frac{G_m}{C_m} V_m - \frac{g_r K_o}{C_m}\dot{x}_r$$

For a frequency-domain solution, let

$$Z_b = M_b iw + R_b + K_b \frac{1}{iw}$$

$$Z_t = M_t iw + R_t + K_t \frac{1}{iw}$$

$$Z_r = M_r iw + R_r + K_r \frac{1}{iw}$$

$$Z_c = R_c + K_c \frac{1}{iw}$$

$$Z_o = \frac{(g_r K_o)^2}{G_m + C_m iw}$$

$$Z_a = g_f g_r \frac{K_o}{iw} \frac{G_m}{G_m + C_m iw}$$

Then the partition impedance and "second-filter" will be

$$Z_p = Z_b + g_c H_c Z_c$$

$$H_c = \frac{Z_r H_t}{(Z_o + g_c Z_c) H_t + Z_r}$$

and

$$H_t = \frac{g_c Z_t}{Z_t + Z_c}$$

If the tectorial membrane is much stiffer to lateral motion than the hair bundles, then the contribution of x_t to x_c may be negligible and the outer-hair cell motility can be modeled by only 2 mechanical degrees of freedom. In this case,

$$x_c = g_c (x_b - x_r)$$

The formulation of the time-domain solution is the same as described above, except that $\ddot{x}_t$ need not be computed. The formulation of the frequency-domain solution is the same except that

$$H_t = g_c$$

12. Time-domain solution (alternate formulation)

It may be more convenient to formulate solution methods in terms of pressure instead of acceleration. For the time-domain solution, we can write

$$[A]\underline{P} = \underline{Q}$$

where $\underline{P}$ includes a corresponding pressure variable for each displacement variable in the state vector $\underline{x}$.

$$\underline{P} = \begin{bmatrix} P_d \\ P_e \\ P_m \\ P_s \\ P_{f_1} \\ P_{f_2} \\ \vdots \\ P_{f_{N-2}} \\ P_{f_{N-1}} \end{bmatrix}$$

First, we express acceleration variables in terms of pressure variables:

$$\ddot{x}_d = \frac{A_d^2}{M_d}\left(S_{d_0} - S_{d_1} - P_d\right)$$

$$\ddot{x}_e = \frac{A_e^2}{M_e}\left(S_{e_0} - S_{e_1} - P_e\right)$$

$$\ddot{x}_m = \frac{A_m^2}{M_m}\left(P_e - S_{m_1} - P_m\right)$$

$$\ddot{x}_s = \frac{A_s^2}{M_s}\left(S_{s_0} - S_{s_1} - P_s\right)$$

$$\ddot{x}_{f_{i+1}} = \frac{A_{c_i}^2}{M_{f_i}}\left(P_{f_i} - P_{f_{i+1}} - S_{f_i}\right)$$

where

$$S_{d_0} = \frac{C_p}{A_d}V_p$$

$$S_{d_1} = \left(\frac{R_p + R_d}{A_d}\right)\frac{\dot{x}_d}{A_d} + \left(\frac{K_d}{A_d}\right)\frac{x_d}{A_d}$$

$$S_{e_0} = \left(\frac{R_a}{A_e}\right)\left(\frac{\dot{x}_d}{A_d} - \frac{\dot{x}_e}{A_e}\right) + \left(\frac{K_a}{A_e}\right)\left(\frac{x_d}{A_d} - \frac{x_e}{A_e}\right)$$

$$S_{e_1} = \left(\frac{R_e}{A_e}\right)\frac{\dot{x}_e}{A_e}$$

$$S_{m_0} = \left(\frac{R_u}{A_u}\right)\frac{\dot{x}_u}{A_u} + \left(\frac{K_u}{A_u}\right)\frac{x_u}{A_u}$$

$$S_{m_1} = \left(\frac{R_m}{A_m}\right)\frac{\dot{x}_m}{A_m} + \left(\frac{K_m}{A_m}\right)\frac{x_m}{A_m}$$

$$S_{s_0} = \left(\frac{R_i}{A_s}\right)\left(\frac{g_m\dot{x}_m}{A_m} - \frac{\dot{x}_s}{A_s}\right) + \left(\frac{K_i}{A_s}\right)\left(\frac{g_m x_m}{A_m} - \frac{x_s}{A_s}\right)$$

$$S_{s_1} = \left(\frac{R_s}{A_s}\right)\frac{\dot{x}_s}{A_s} + \left(\frac{K_s}{A_s}\right)\frac{x_s}{A_s}$$

$$S_{f_i} = \left(\frac{R_{f_i}}{A_{c_i}}\right)\frac{\dot{x}_{f_{i+1}}}{A_{c_i}}$$

and S_{p_i} depends on the details of the micromechanics. We derive $[A]$ and $\underline{Q}$ by considering equations for pressure at each stage in the model.

At the earphone:

$$\ddot{x}_d = \frac{A_d}{A_e}\left(\ddot{x}_a + \ddot{x}_e\right)$$

$$P_d = \frac{A_e}{A_d} S_{e_0}$$

At the eardrum:

$$\ddot{x}_e = \ddot{x}_a + \ddot{x}_e$$

$$\frac{A_e^2}{M_e}\left(S_{e_0} - S_{e_1} - P_e\right) = \frac{A_u^2}{M_u}\left(P_e - S_{m_0}\right) + \frac{A_m^2}{M_m}\left(P_e - S_{m_1} - P_m\right)$$

$$\left(\frac{A_e^2}{M_e} + \frac{A_u^2}{M_u} + \frac{A_m^2}{M_m}\right)P_e - \frac{A_m^2}{M_m}P_m = \frac{A_e^2}{M_e}\left(S_{e_0} - S_{e_1}\right) + \frac{A_u^2}{M_u}S_{m_0} + \frac{A_m^2}{M_m}S_{m_1}$$

At the malleus:

$$\ddot{x}_m = \frac{A_m}{g_m A_s}\left(\ddot{x}_i + \ddot{x}_s\right)$$

$$P_m = \frac{g_m A_s}{A_m} S_{s_0}$$

At the stapes:

$$\ddot{x}_s = \ddot{x}_{p_0} + \ddot{x}_{f_0}$$

$$\frac{A_s^2}{M_s}\left(S_{s_0} - S_{s_1} - P_s\right) = \frac{A_{p_0}^2}{M_{p_0}}\left(P_s - S_{p_0}\right) + \frac{A_{c_0}^2}{M_{f_0}}\left(P_{f_0} - P_{f_1} - S_{f_0}\right)$$

$$\left(\frac{A_s^2}{M_s}+\frac{A_{p_0}^2}{M_{p_0}}+\frac{A_{c_0}^2}{M_{f_0}}\right)P_s-\frac{A_{c_0}^2}{M_{f_0}}P_{f_1}=\frac{A_s^2}{M_s}\left(S_{s_0}-S_{s_1}\right)+\frac{A_{p_0}^2}{M_{p_0}}S_{p_0}+\frac{A_{c_0}^2}{M_{c_0}}S_{f_0}$$

In the cochlea:

$$\ddot{x}_{f_i}=\ddot{x}_{p_i}+\ddot{x}_{f_{i+1}}$$

$$\frac{A_{c_{i-1}}^2}{M_{f_{i-1}}}\left(P_{f_{i-1}}-P_{f_i}-S_{f_{i-1}}\right)=\frac{A_{p_i}^2}{M_{p_i}}\left(P_{f_i}-S_{p_i}\right)+\frac{A_{c_i}^2}{M_{f_i}}\left(P_{f_i}-P_{f_{i+1}}-S_{f_i}\right)$$

$$\left(\frac{A_{c_{i-1}}^2}{M_{f_{i-1}}}+\frac{A_{p_i}^2}{M_{p_i}}+\frac{A_{c_i}^2}{M_{f_i}}\right)P_{f_i}-\frac{A_{c_{i-1}}^2}{M_{f_{i-1}}}P_{f_{i-1}}-\frac{A_{c_i}^2}{M_{c_i}}P_{f_{i+1}}=\frac{A_{c_i}^2}{M_{c_i}}S_{f_i}-\frac{A_{c_{i-1}}^2}{M_{f_{i-1}}}S_{f_{i-1}}+\frac{A_{p_i}^2}{M_{p_i}}S_{p_i}$$

Now we can write

$$[A]=\begin{bmatrix} a_d & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_e & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_m & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & a_s & -a_{f_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -a_{f_0} & a_{p_1} & -a_{f_1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{f_1} & a_{p_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{p_{N-2}} & -a_{f_{N-2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -a_{f_{N-2}} & a_{p_{N-1}} \end{bmatrix}$$

where $a_d=\frac{A_d^2}{M_d}$, $a_e=\left(\frac{A_e^2}{M_e}+\frac{A_u^2}{M_u}+\frac{A_m^2}{M_m}\right)$, $a_m=\frac{A_m^2}{M_m}$, $a_s=\left(\frac{A_{c_0}^2}{M_{f_0}}+\frac{A_s^2}{M_s}+\frac{A_{p_0}^2}{M_{p_0}}\right)$, $a_{f_i}=\frac{A_{c_i}^2}{M_{f_i}}$,

$a_{p_i}=\left(\frac{A_{c_i}^2}{M_{c_i}}+\frac{A_{c_{i-1}}^2}{M_{c_{i-1}}}+\frac{A_{p_i}^2}{M_{p_i}}\right)$ and

$$
\underline{Q}=\begin{bmatrix}
\frac{A_e A_d}{M_d} S_{d_1} \\
\frac{A_e^2}{M_e}\left(S_{e_0}-S_{e_1}\right)+\frac{A_u^2}{M_u} S_{m_0}+\frac{A_m^2}{M_m} S_{m_1}+\frac{g_m A_m A_s}{M_m} S_{s_0} \\
\frac{g_m A_m A_s}{M_m} S_{s_0} \\
\frac{A_s^2}{M_s}\left(S_{s_0}-S_{s_1}\right)+\frac{A_{p_0}^2}{M_{p_0}} S_{p_0}+\frac{A_{c_0}^2}{M_{f_0}} S_{f_0} \\
\frac{A_{c_1}^2}{M_{f_1}} S_{f_1}-\frac{A_{c_0}^2}{M_{c_0}} S_{f_0}+\frac{A_{p_1}^2}{M_{p_1}} S_{p_1} \\
\frac{A_{c_2}^2}{M_{f_2}} S_{f_2}-\frac{A_{c_1}^2}{M_{c_1}} S_{f_1}+\frac{A_{p_2}^2}{M_{p_2}} S_{p_2} \\
\vdots \\
\frac{A_{c_{N-2}}^2}{M_{f_{N-2}}} S_{f_{N-2}}-\frac{A_{c_{N-3}}^2}{M_{c_{N-3}}} S_{f_{N-3}}+\frac{A_{p_{N-2}}^2}{M_{p_{N-2}}} S_{p_{N-2}} \\
\frac{A_{c_{N-1}}^2}{M_{f_{N-1}}} S_{f_{N-1}}-\frac{A_{c_{N-2}}^2}{M_{c_{N-2}}} S_{f_{N-2}}+\frac{A_{p_{N-1}}^2}{M_{p_{N-1}}} S_{p_{N-1}}
\end{bmatrix}
$$

Once the pressure $\underline{P}$ has been obtained, we can compute the acceleration values needed for the time-domain solution from the equations above.

13. Frequency-domain solution (alternate formulation)

In the frequency domain, we can solve for the pressure in the equation

$$[A]\underline{P} = \underline{Q}$$

where

$$\underline{P} = \begin{bmatrix} P_d \\ P_e \\ P_m \\ P_s \\ P_{f_1} \\ P_{f_2} \\ \vdots \\ P_{f_{N-2}} \\ P_{f_{N-1}} \end{bmatrix} \qquad \underline{Q} = \begin{bmatrix} \frac{A_d C_p}{Z_a} V_p \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

First, we define a set of mechanical impedance functions:

$$Z_d = M_d iw + (R_d + R_p) + K_d \frac{1}{iw}$$

$$Z_a = R_a + K_a \frac{1}{iw}$$

$$Z_e = M_e iw + R_e$$

$$Z_m = M_m iw + R_m + K_m \frac{1}{iw}$$

$$Z_u = M_u iw + R_u + K_u \frac{1}{iw}$$

$$Z_i = R_i + K_i \frac{1}{iw}$$

$$Z_s = M_s iw + R_s + K_s \frac{1}{iw}$$

$$Z_{f_i} = M_{f_i} iw + R_{f_i}$$

$$Z_{p_i} = Z_{b_i} + g_{c_i} H_{c_i} Z_{c_i}$$

The transfer function H_{c_i} depends on the details of the micromechanics. Next, we express velocity variables in terms of pressures

$$\dot{x}_d = \frac{A_d}{Z_d}\left(C_p V_p - A_d P_d\right)$$

$$\dot{x}_e = \frac{A_e}{Z_e}\left(A_d P_d - A_e P_e\right)$$

$$\dot{x}_m = \frac{A_m^2}{Z_m}\left(P_e - P_m\right)$$

$$\dot{x}_s = \frac{A_s}{Z_s}\left(\frac{A_m}{g_m}P_m - A_s P_s\right)$$

$$\dot{x}_{f_{i+1}} = \frac{A_{c_i}^2}{Z_{f_i}}\left(P_{f_i} - P_{f_{i+1}}\right)$$

$$\dot{x}_{p_i} = \frac{A_{p_i}^2}{Z_{p_i}}P_{f_i}$$

Now, we can express the relation between the velocity variables in terms of pressure variables.

At the earphone:

$$\dot{x}_d = \frac{A_d}{A_e}\left(\dot{x}_a + \dot{x}_e\right)$$

$$\frac{A_d}{Z_d}\left(C_p V_p - A_d P_d\right) = \frac{A_d}{Z_a}\left(A_d P_d\right) + \frac{A_d}{Z_e}\left(A_d P_d - A_e P_e\right)$$

$$\left(\frac{A_d^2}{Z_d} + \frac{A_d^2}{Z_a} + \frac{A_d^2}{Z_e}\right)P_d = \frac{A_d C_p}{Z_d}V_p + \frac{A_d A_e}{Z_e}P_e$$

At the eardrum:

$$\dot{x}_e = \dot{x}_u + \dot{x}_m$$

$$\frac{A_e}{Z_e}\left(A_d P_d - A_e P_e\right) = \frac{A_u}{Z_u}\left(A_u P_u\right) + \frac{A_m}{Z_m}\left(A_m P_e - A_m P_m\right)$$

$$\left(\frac{A_e^2}{Z_e} + \frac{A_u^2}{Z_u} + \frac{A_m^2}{Z_m}\right)P_e = \frac{A_e A_d}{Z_e}P_d + \frac{A_m^2}{Z_m}P_s$$

At the malleus:

$$\dot{x}_m = \frac{A_m}{g_m A_s}\left(\dot{x}_i + \dot{x}_s\right)$$

$$\frac{A_m}{Z_m}\left(A_m P_e - A_m P_m\right) = \frac{A_m}{g_m Z_i}\left(\frac{A_m}{g_m}P_m\right) + \frac{A_m}{g_m Z_s}\left(\frac{A_m}{g_m}P_m - A_s P_s\right)$$

$$\left(\frac{A_m^2}{Z_m} + \frac{A_m^2}{g_m^2 Z_i} + \frac{A_m^2}{g_m^2 Z_s}\right)P_m = \frac{A_m^2}{Z_m}P_e + \frac{A_m A_s}{g_m Z_s}P_s$$

At the stapes:

$$\dot{x}_s = \dot{x}_{p_0} + \dot{x}_{f_0}$$

$$\frac{A_s}{Z_s}\left(\frac{A_m}{g_m}P_m - A_s P_s\right) = \frac{A_{p_o}^2}{Z_{p_o}}P_s + \frac{A_{c_0}^2}{Z_{f_0}}\left(P_s - P_{f_1}\right)$$

$$\left(\frac{A_s^2}{Z_s}+\frac{A_{p_0}^2}{Z_{p_0}}+\frac{A_{c_0}^2}{Z_{c_0}}\right)P_s=\frac{A_m A_s}{g_m Z_s}P_m+\frac{A_{c_0}^2}{Z_{f_0}}P_{f_1}$$

In the cochlea:

$$\dot{x}_{f_i}=\dot{x}_{p_i}+\dot{x}_{f_{i+1}}$$

$$\frac{A_{c_{i-1}}^2}{Z_{f_{i-1}}}\left(P_{f_{i-1}}-P_{f_i}\right)=\frac{A_{p_i}^2}{Z_{p_i}}P_{f_i}+\frac{A_{c_i}^2}{Z_{f_i}}\left(P_{f_i}-P_{f_{i+1}}\right)$$

$$\left(\frac{A_{c_{i-1}}^2}{Z_{f_{i-1}}}+\frac{A_{p_i}^2}{Z_{p_i}}+\frac{A_{c_i}^2}{Z_{c_i}}\right)P_{f_i}=\frac{A_{c_{i-1}}^2}{Z_{f_{i-1}}}P_{f_i}+\frac{A_{c_i}^2}{Z_{f_i}}P_{f_{i+1}}$$

From these equations we can determine the components of the matrix

$$[A]=\begin{bmatrix} a_d & -a_a & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -a_a & a_e & -a_b & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -a_b & a_m & -a_u & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -a_u & a_s & -a_{f_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -a_{f_0} & a_{p_1} & -a_{f_1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{f_1} & a_{p_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{p_{N-2}} & -a_{f_{N-2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -a_{f_{N-2}} & a_{p_{N-1}} \end{bmatrix}$$

where $a_d=\left(\frac{A_d^2}{Z_d}+\frac{A_d^2}{Z_a}+\frac{A_d^2}{Z_e}\right)$, $a_a=\frac{A_d A_e}{Z_e}$, $a_e=\left(\frac{A_e^2}{Z_e}+\frac{A_u^2}{Z_u}+\frac{A_m^2}{Z_m}\right)$, $a_b=\frac{A_m^2}{Z_m}$, $a_m=\left(\frac{A_m^2}{Z_m}+\frac{A_m^2}{g_m Z_i}+\frac{A_m^2}{g_m Z_s}\right)$, $a_u=\frac{A_m A_s}{g_m Z_s}$, $a_s=\left(\frac{A_s^2}{Z_s}+\frac{A_{p_0}^2}{Z_{p_0}}+\frac{A_{c_0}^2}{Z_{f_0}}\right)$, $a_{f_i}=\frac{A_{c_i}^2}{Z_{f_i}}$, $a_{p_i}=\left(\frac{A_{c_{i-1}}^2}{Z_{c_{i-1}}}+\frac{A_{p_i}^2}{Z_{p_i}}+\frac{A_{c_i}^2}{Z_{c_i}}\right)$. From the pressure, we can compute the velocity from the equations above.

14. Transmission Matrices for Cochlear Macromechanics

The relation between the input and output variable of each stage of the peripheral auditory system model can be described in the frequency domain in terms of transmission (ABCD) matices.

For the acoustic coupler:

$$\begin{bmatrix} P_d \\ \dot{x}_d \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P_e \\ \dot{x}_e \end{bmatrix}$$

where

$$A = \frac{A_e}{A_d} \qquad B = \frac{Z_e}{A_d A_e}$$

$$C = \frac{A_d A_e}{Z_a} \qquad D = \frac{A_d}{A_a}\left(1 + \frac{Z_e}{Z_a}\right)$$

$$Z_a = R_a + K_a \frac{1}{iw} \qquad Z_e = M_e iw + R_e$$

The middle ear has two stages, from the eadrum to the incudo-malleolar joint and from there to the stapes. For the first middle ear stage

$$\begin{bmatrix} P_e \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{Z_m}{A_m^2} \\ \dfrac{A_u^2}{Z_u} & 1 + \dfrac{A_u^2 Z_m}{A_m^2 Z_u} \end{bmatrix} \begin{bmatrix} P_m \\ \dot{x}_m \end{bmatrix}$$

and the second stage

$$\begin{bmatrix} P_m \\ \dot{x}_m \end{bmatrix} = \begin{bmatrix} \dfrac{g_m A_s}{A_m} & \dfrac{g_m Z_s}{A_m A_s} \\ \dfrac{A_m A_s}{g_m Z_i} & \dfrac{A_m}{g_m A_s}\left(1 + \dfrac{Z_s}{Z_i}\right) \end{bmatrix} \begin{bmatrix} P_s \\ \dot{x}_s \end{bmatrix}$$

After multiplying these two stages together

$$\begin{bmatrix} P_e \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P_s \\ \dot{x}_s \end{bmatrix}$$

where

$$A = \frac{g_m A_s}{A_m}\left(1 + \frac{Z_m}{g_m^2 Z_i}\right)$$

$$B = \frac{g_m Z_s}{A_m A_s}\left(1 + \frac{Z_m}{g_m^2 Z_s} + \frac{Z_m}{g_m^2 A_u^2 Z_i}\right)$$

$$C = \frac{g_m A_s A_u^2}{A_m Z_u}\left(1 + \frac{A_m^2 Z_u}{g_m^2 A_u^2 Z_i} + \frac{Z_m}{g_m^2 Z_i}\right)$$

$$D = \frac{A_m}{g_m A_s}\left(1 + \frac{Z_s}{Z_i} + \frac{A_u^2 Z_m}{A_m^2 Z_u} + \frac{A_u^2 Z_m Z_s}{A_m^2 Z_u Z_i} + \frac{g_m^2 A_u^2 Z_s}{A_m^2 Z_u}\right)$$

$$Z_m = M_m iw + R_m + K_m \frac{1}{iw}$$

$$Z_u = M_u iw + R_u + K_u \frac{1}{iw}$$

$$Z_s = M_s iw + R_s + K_s \frac{1}{iw}$$

$$Z_i = R_i + K_i \frac{1}{iw}$$

For a radial cross-section of the cochlea

$$\begin{bmatrix} P_{f_i} \\ \dot{x}_{f_i} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P_{f_{i+1}} \\ \dot{x}_{f_{i+1}} \end{bmatrix}$$

where

$$A = 1 \qquad B = \frac{Z_{f_i}}{A_{c_i}^2}$$

$$C = \frac{A_{p_i}^2}{Z_{p_i}} \qquad D = 1 + \frac{A_{p_i}^2 Z_{f_i}}{A_{p_i}^2 Z_{p_i}}$$

$$Z_{f_i} = M_{f_i} iw + R_{f_i} \qquad Z_{p_i} = Z_{b_i} + g_{c_i} Z_{c_i} H_{c_i}$$

15. Transmission Matrix for Cochlear Micromechanics

The effect of the micromechanics on the transmission of vibrations from the basilar membrane to the hair bundles of the inner and outer hair cells can be described in terms of a transmission (ABCD) matrix. To derive the components of this matrix, we start by writing the equations of motion in the frequency domain

$$Z_b \dot{x}_b = F_b - g_c F_c$$

$$Z_t \dot{x}_t = F_c$$

$$Z_r \dot{x}_r = g_c F_c + F_o$$

where

$$F_b = -A_p P_f$$

$$F_c = Z_c \dot{x}_c$$

$$F_o = Z_o \dot{x}_c$$

$$Z_b = M_b iw + R_b + K_b \frac{1}{iw}$$

$$Z_t = M_t iw + R_t + K_t \frac{1}{iw}$$

$$Z_r = M_r iw + R_r + K_r \frac{1}{iw}$$

$$Z_c = R_c + K_c \frac{1}{iw}$$

$$Z_o = g_f g_r \frac{K_o}{iw}\left(\frac{G_m}{G_m + C_m iw}\right)$$

we want to rewrite these equations in the form

$$\begin{bmatrix} F_b \\ \dot{x}_b \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} F_c \\ \dot{x}_c \end{bmatrix}$$

From the equations above, we have

$$A = g_c + Z_b\left(\frac{1}{g_c Z_t} + \frac{g_c}{Z_r}\right)$$

$$B = Z_b\left(\frac{1}{g_c} + \frac{Z_o}{Z_r}\right)$$

$$C = \left(\frac{1}{g_c Z_t} + \frac{g_c}{Z_r}\right)$$

$$D = \left(\frac{1}{g_c} + \frac{Z_o}{Z_r}\right)$$

16. Longitudinal Stiffness

Consider the effect of longitudinal stiffness on cochlear micromechanics. Displacement of the basilar membrane will be opposed by its connections to adjacent radial cross-sections.

$$M_{b_i}\ddot{x}_{b_i} + R_{b_i}\dot{x}_{b_i} + K_{b_i}x_{b_i} + g_{c_i}R_{c_i}\dot{x}_{c_i} + g_{c_i}K_{c_i}x_{c_i} + K_{l_i}\left(x_{b_i} - x_{b_{i+1}}\right) + K_{l_{i-1}}\left(x_{b_i} - x_{b_{i-1}}\right) = -A_{p_i}P_{f_i}$$

For time-domain solutions, the only effect of longitudinal stiffness is on S_{p_i}

$$S_{p_i} = \frac{-1}{A_{p_i}}\left[R_{b_i}\dot{x}_{b_i} + K_{b_i}x_{b_i} + g_{c_i}R_{c_i}\dot{x}_{c_i} + g_{c_i}K_{c_i}x_{c_i} + K_{l_i}\left(x_{b_i} - x_{b_{i+1}}\right) + K_{l_{i-1}}\left(x_{b_i} - x_{b_{i-1}}\right)\right]$$

For frequency-domain solutions, the effect of longitudinal stiffness is more difficult to incorporate. If we let

$$Z_{b_i} = M_{b_i}iw + R_{b_i} + K_{b_i}\frac{1}{iw}$$

$$Z_{c_i} = R_{c_i} + K_{c_i}\frac{1}{iw}$$

$$Z_{l_i} = K_{l_i}\frac{1}{iw}$$

Then the equation of motion becomes

$$\left(Z_{b_i} + g_{c_i}Z_{c_i}H_{c_i}\right)\dot{x}_{b_i} + Z_{l_i}\left(\dot{x}_{b_i} - \dot{x}_{b_{i+1}}\right) + Z_{l_{i-1}}\left(\dot{x}_{b_i} - \dot{x}_{b_{i-1}}\right) = -A_{p_i}P_{f_i}$$

Let

$$Z_{p_i} = Z_{b_i} + g_{c_i}Z_{c_i}H_{c_i}$$

$$a_i = \frac{Z_{l_i}}{Z_{p_i}} \qquad b_i = \frac{Z_{l_{i-1}}}{Z_{p_i}}$$

$$y_{f_i} = \frac{A_{c_i}^2}{Z_{f_i}} \qquad y_{p_i} = \frac{A_{p_i}^2}{Z_{p_i}}$$

then for $0 < i < N-1$

$$y_{p_i}P_{f_i} = \dot{x}_{p_i} - a_i\dot{x}_{p_{i+1}} - b_i\dot{x}_{p_{i-1}}$$

$$\dot{x}_{p_i} = y_{f_{i-1}}P_{f_{i-1}} + y_{f_i}P_{f_{i+1}} - \left(y_{f_{i-1}} + y_{f_i}\right)P_{f_i}$$

$$\left[y_{p_i} + y_{f_i}(1+a_i) + y_{f_{i-1}}(1+b_i)\right]P_{f_i} = \left[y_{f_{i-1}} + b_i\left(y_{f_{i-1}} + y_{f_{i-2}}\right)\right]P_{f_{i-1}} + \left[y_{f_i} + a_i\left(y_{f_i} + y_{f_{i+1}}\right)\right]P_{f_{i+1}}$$
$$- b_i y_{f_{i-2}}P_{f_{i-2}} - a_i y_{f_{i+1}}P_{f_{i+2}}$$

At the stapes boundary, let $y_s = \frac{A_s^2}{Z_s}$, so that

$$y_{p_0}P_s = \dot{x}_s - a_0\dot{x}_{p_1}$$

$$\dot{x}_s = y_sP_m + y_{f_0}P_{f_1} - \left(y_s + y_{f_0}\right)P_s$$

$$\left[y_{p_0} + y_{f_0}(1+a_0)\right]P_s = y_sP_m + \left[y_{f_0} + a_0\left(y_{f_0} + y_{f_1}\right)\right]P_{f_1} - a_0P_{f_2}$$

At the helicotrema boundary, let $y_h = \dfrac{A_h^2}{Z_h}$, so that

$$y_h P_{f_{N-1}} = \dot{x}_{p_{N-1}} - B_{N-1}\dot{x}_{p_{N-2}}$$

$$\dot{x}_{p_{N-1}} = y_{f_{N-2}} P_{f_{N-2}} + y_{f_{N-2}} P_{f_{N-1}}$$

$$\left[y_h + y_{f_{N-2}}\left(1 + b_{N-1}\right)\right] P_{f_{N-1}} = \left[y_{f_{N-2}} + b_{N-1}\left(y_{f_{N-3}} + y_{f_{N-2}}\right)\right] P_{f_{N-2}} - b_{N-1} y_{f_{N-3}} P_{f_{N-3}}$$